Example 1

In one of the experiments, the experimenter is interested in making comparisons among 7 treatments and there are 28 experimental units available. These 28 experimental units are arranged in a Youden Square design with 4 rows and 7 columns with one observation per row.

2(4.00)	3(5.30)	4(1.0)	5(16.9)	6(16.9)	7(10.3)	1(294)
7(17.5)	1(222)	2(12.20)	3(15.5)	4(11)	5(26.5)	6(27.2)
6(37)	7(26)	1(310)	2(22.7)	3(24.2)	4(21.4)	5(31.2)
5(46.8)	6(44.2)	7(34.3)	1(282)	2(33.7)	3(33.7)	4(30.50)

Solution

$$t = 7, blocks = 7, r = k = 4, \lambda = \frac{r(k-1)}{t-1} = 2$$

$$SS_T = \sum_i \sum_j \sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$$

$$Q_i = y_{i.} - \frac{1}{k} \sum_{j=1}^{b} n_{ij} y_{.j}$$

$$SS_{Treatments} = \frac{k \sum_{i=1}^{a} Q_i^2}{\lambda a}$$

SSrows and SS columns are computed normally as under the Latin Square Design.

Rows	3	3319.006	1106.33	4.89
Columns	6	27595.41	4599.23	20.31
Treatment	6	195027.72	32504.62	143.56
Error	12	2717.08	226.42	
Total	27	228659.22		

Exercise

- Discuss the following types of lattice designs
  - Rectangular Lattice Designs
  - Cubic Lattice Designs

Lattice Designs

Consider a balanced incomplete block design with k^2 treatments arranged in $b = k(k+1)$ blocks with k runs per block and $r = k+1$ replicates. Such a design is called a **balanced lattice**

The analysis of variance for the balanced lattice design proceeds like that for a balanced incomplete block design, except that a sum of squares for replicates is computed and removed from the sum of squares for blocks.

Replicates will have k degrees of freedom and blocks will have $k^2 - 1$ degrees of freedom.

Two replicates of a design for k^2 in $2k$ blocks of k runs are called a **simple lattice**

Statistical Analysis

We will adopt the following notation

- t denote the total number of treatments
- k denote the number of units per block or block size
- s denotes the number of blocks per replication which is equal to k
- r denotes the number of replications

Let $y_{ij(l)}$ denote the response value of the j^{th} treatment in the l^{th} block within i^{th} replication $i = 1, 2, ..., k+1, j = 1, 2, ..., k^2, l = 1, 2, ..., rk$.

The model is given as

$$y_{ij(l)} = \mu + \pi_i + \beta_{i(l)} + \tau_j + \epsilon_{ij(l)}$$

where $\mu, \pi_i, \beta_{i(l)}$ and τ_j represent the effect of the mean, the replicate, the icnomplete block and the treatment respectively.

Various ANOVA Sum of Squares are given by:

Total Sum of Squares

$$SSTot = \sum y_{ij(l)}^2 - \frac{(\sum y_{ij(l)})^2}{rk^2}$$

Unadujsted treatment sum of squares

$$SSTrt_U = \frac{\sum T_j^2}{r} - \frac{(\sum y_{ij(l)})^2}{rk^2}$$

Replication sum of squares

$$SSR = \frac{\sum R_i^2}{k^2} - \frac{(\sum y_{ij(l)})^2}{rk^2}$$

where R_i is the sum of replications in replication i

Adjusted block sum of squares SSB_{adj}

B_j denote the sum of block totals for the blocks with treatment j

T_j denote the total of the j^{th} treatment total from all replications

W_j denote the weight for the j^{th} treatment which is used for adjusted for block

$$W_j = kT_j - (k+1)B_j + G$$

where $G = \sum y_{ij(l)}$

The $SSB_{Adj} = \frac{\sum W_j^2}{k^3(k+1)}$

Intra-block error sum of squares

$$SSE = SST - SSR - SSTrt_U - SSB_{Adj}$$

Source	df	SS	MSS
Replication	$r-1$	SSR	MSR
Treatment(unadjusted)	k^2-1	SST_{trU}	MST
Block (adjusted)	k^2-1	SSB_{adj}	MSB
Error	$(k^2-1)(k-1)$	SSE	MSE
Total	rk^2-1	SST	

The ANOVA for the balanced square lattice design

It is important to note that the mean square of the unadjusted treatment cannot be used for testing against the mean square of intra-block error because the mean square of unadjusted treatment still contains block effects.

The adjusted treatment sum of squares is defined as:

$$SST_{adj} = \frac{\sum(T_j + \mu W_j)^2}{r} - \frac{(\sum y_{ij(l)})^2}{rk^2}$$

Where μ is the adjustment factor for error and treatment means defined as:

$$\mu = \frac{MSB_{Adj} - MSE}{k^2 MSB_{Adj}}$$

Example

Analyze the following balanced square lattice design at 5% level of significance.

Block	Replication I			Total	Block	Replication II			Total
1	(1)	(2)	(3)		4	(3)	(4)	(8)	
	2.20	1.84	2.18	6.22		1.71	1.57	1.13	4.41
2	(4)	(5)	(6)		5	(2)	(6)	(7)	
	2.05	0.85	1.86	4.76		1.76	2.16	1.80	5.72
3	(7)	(8)	(9)		6	(1)	(5)	(9)	
	0.73	1.60	1.76	4.09		1.81	1.16	1.10	4.08
Block	Replication III			Total	Block	Replication IV			Total
7	(1)	(4)	(7)		10	(3)	(5)	(7)	
	1.19	1.20	1.15	3.54		2.04	0.93	1.78	4.75
8	(2)	(5)	(8)		11	(2)	(4)	(9)	
	2.26	1.07	1.45	4.78		1.50	1.60	1.42	4.52
9	(3)	(6)	(9)		12	(1)	(6)	(8)	
	2.12	2.03	1.63	5.78		1.77	1.57	1.43	4.77

Solution

$$SST = (2.20^2 + 1.84^2 + ... + 1.43^2) - CF = 97.55 - 91.58 = 5.97$$

$$SSTrt_U = \frac{\sum T_j^2}{r} - CF = \frac{6.97^2 + ... + 5.92^2}{4} - 91.58 = 3.23$$

$$SSR = \frac{15.07^2 + ... + 14.04^2}{3^2} - 91.58 = 91.66 - 91.58 = 0.08$$

Then, to find the adjusted sum of squares for blocks, (1) the sum of block totals for block with the jth treatment, $B_{j,(2)}$ the total of treatment j for all replications, T_j and (3) the weights for treatment j, W_j are required. For example, for treatment 7,

$$T_7 = 0.73 + 1.80 + 1.15 + 1.78 = 5.46$$

$$B_7 = 4.09 + 5.72 + 3.54 + 4.75 = 18.10$$

$$W_7 = 3(5.46) - 4(18.10) + G = 1.40$$

G is grand total

The values of B_j, T_j, W_j are summarized

Trt j	T_j	B_j	$W_j = kT_j - (k+1)B_j + G$
1	6.97	18.61	3.89
2	7.36	21.24	-5.46
3	8.05	21.16	-3.07
4	6.42	17.23	7.76
5	4.01	18.37	-4.03
6	7.62	21.03	-3.84
7	5.46	18.10	1.40
8	5.61	18.05	2.05
9	5.92	18.47	1.30

$$SSB_{Adj} = \frac{\sum W_j^2}{k^3(k+1)} = \frac{3.89^2 + ... + 1.30^2}{3^2(3+1)} = \frac{153.43}{108} = 1.42$$

The intra-block error sum of squares

$$SSB_{Adj} = SST - SSR - SSTrt_U - SSB_{Adj} = 5.97 - 0.08 - 3.23 - 1.42 = 1.24$$

Source of variation	Degree of freedom	Sum of squares	Mean squares
Replication	$4 - 1 = 3$	0.08	0.026
Treatment(Unadjusted)	$3^2 - 1 = 8$	3.23	0.404
Block within replication(Adj)	$3^2 - 1 = 8$	1.42	$0.178 = E_b$
Intra-block Error	$(3^2 - 1)(3 - 1) = 16$	1.24	$0.077 = E_e$
Total	$4(3^2) - 1 = 35$	5.97	

The adjustment factor

$$\mu = \frac{E_b - E_e}{k^2 E_b} = \frac{0.178 - 0.078}{3^2(0.178)} = 0.062$$

The adjusted treatment total $T_j' = T_j + \mu W_j$

Adjusted treatment totals								
T_1'	T_2'	T_3'	T_4'	T_5'	T_6'	T_7'	T_8'	T_9'
7.21	7.01	7.85	6.90	3.76	7.38	5.55	5.74	6.00

$$SST_{Adj} = \frac{\sum (T_j')^2}{r} - G = 3.16$$

The effective error mean squares

$$E_e' = E_e[1 + (r-1)\mu] = 0.078[1 + 3(0.062)] = 0.092$$

The F-ration equals

$$F = \frac{3.16/8}{0.092} = 4.30$$